

## A FINITENESS THEOREM FOR NEGATIVELY CURVED MANIFOLDS

KENJI FUKAYA

### 0. Introduction

The purpose of this paper is to investigate the topological structure of negatively curved manifolds. We show the finiteness of the number of diffeomorphism classes containing complete (but not necessarily compact) negatively curved manifolds with bounded volumes and curvatures. This result is related to works of Wang, Cheeger, Gromov, and others. Hence we first give a rough summary of some of their works.

Wang investigated the case of locally symmetric spaces. Let  $X$  be a simply connected symmetric space of noncompact type without factors of dimension smaller than 4. Denote by  $\text{Iso}(X)$  the group of all selfisometries of  $X$ . Wang considered subgroups  $\Gamma$  of  $\text{Iso}(X)$  acting on  $X$  effectively and properly discontinuously. He proved that, for each positive number  $V$ , there exist only a finite number of conjugacy classes of  $\Gamma$  satisfying  $\text{Vol}(\Gamma \backslash X) < V$ .

On the other hand, in [5], Gromov proved a finiteness theorem for Riemannian manifolds  $M$  satisfying

$$(*) \quad \text{Vol}(M) \leq V,$$

$$(**) \quad 0 > \text{sectional curvature} \geq -1.$$

Namely, he proved that, for each positive number  $V$  and for each positive integer  $n$  greater than 3, there exist only a finite number of diffeomorphism classes containing compact  $n$ -dimensional Riemannian manifolds  $M$  satisfying  $(*)$  and  $(**)$ .

In the case of locally symmetric spaces, Wang's result is stronger than Gromov's. We reach the problem below.

Consider pairs  $(X, \Gamma)$ , where  $X$  is a simply connected and complete Riemannian manifold and  $\Gamma$  is a group of isometries of  $X$  acting effectively and

properly discontinuously. We say  $(X, \Gamma)$  and  $(X', \Gamma')$  are equivariantly diffeomorphic to each other if there exist an isomorphism  $\Phi: \Gamma \rightarrow \Gamma'$  and a diffeomorphism  $f: X \rightarrow X'$  such that  $f(\gamma(x)) = \Phi(\gamma)(f(x))$  holds for every  $\gamma \in \Gamma$  and  $x \in X$ .

**Problem.** Let  $n \neq 2, 3$  be a positive integer and  $V$  a positive number. Then there exist only a finite number of equivariant diffeomorphism classes containing  $(X, \Gamma)$  such that

$$\begin{aligned} 0 > \text{sectional curvature of } X &\geq -1, \\ \text{Vol}(\Gamma \backslash X) &\leq V. \end{aligned}$$

Gromov's theorem, described above, gives an affirmative answer to the above problem in the case when the following conditions are satisfied:

- (a)  $\Gamma \backslash X$  is compact.
- (b)  $\Gamma$  acts freely on  $X$ .

One of the main theorems of this paper is a generalization of Gromov's theorem to the case when (a) is not necessarily satisfied.

**Theorem I.** *Let  $V$  be a positive number and let  $n$  be a positive integer with  $n \neq 3, 4$  (resp.  $n = 4$ ). Then there exist only a finite number of diffeomorphism classes (resp. homotopy types) containing  $n$ -dimensional Riemannian manifolds  $M$  satisfying the following conditions:*

- (1)  $M$  has negative curvature or satisfies the visibility axiom of Eberlein & O'Neill [3].
- (2) sectional curvature of  $M \geq -1$ .
- (3)  $\text{Vol}(M) \leq V$ .

We cannot replace Condition (2) by " $M$  has nonpositive curvature". A counterexample is given by §5.

Gromov used Cheeger's finiteness theorem (Theorem 3.1 in [2]) in the proof of his theorem. To apply Cheeger's theorem directly, we have to restrict ourselves to compact manifolds. Hence we need a noncompact version of Cheeger's theorem in order to prove Theorem I. But it seems difficult to obtain a noncompact version of Cheeger's theorem itself. Thus we divide the argument in Cheeger [2] into two parts and generalize each of them. Our generalization of the first part is the following.

**Theorem A.** *For each positive integer  $n$  and positive numbers  $a$  and  $D$ , there exists a positive number  $\varepsilon_1(a, D)$  such that the following holds:*

*Suppose that  $N$  is a  $n$ -dimensional Riemannian manifold whose sectional curvature is not smaller than  $-1$ , and that there is a closed geodesic  $l$  whose length is smaller than  $\varepsilon_1(a, D)$ . Then, for each point  $p$  on  $l$ , we have*

$$\text{Vol}(\{x \in N \mid d(x, p) \leq D\}) \leq a.$$

Theorem A can be proved in exactly the same way as Cheeger [2, Corollary 2-2].

Our generalization of the second part is the following.

**Theorem 1-1.** *Let  $a, b, c$  and  $V$  be positive numbers with  $a < b < c$ , and let  $n$  be a positive integer not equal to 3 or 4 (resp.  $n = 3$  or 4). Then there exist only a finite number of diffeomorphism classes (resp. homotopy types) containing  $n$ -dimensional Riemannian manifolds  $N$  satisfying the following conditions:*

(1)  $-1 \leq \text{sectional curvature} \leq 1$ .

(2)  $\text{Vol}(N) \leq V$ .

(3) *There exist an open subset  $N'$  of  $N$  and a PL-homeomorphism  $\Phi: N - N' \rightarrow \partial N' \times [0, 1)$  which have the following properties:*

(i)  $\partial N'$  is a codimension-1 PL-submanifold of  $N$ .

(ii) *For an element  $p$  of  $\partial N'$  the injectivity radius of  $N$  at  $p$  is greater than  $c$ .*

(iii) *For an element  $p$  of  $\Phi^{-1}(\partial N' \times \{1/2\})$ , the injectivity radius of  $N$  at  $p$  is greater than  $a$  and smaller than  $b$ .*

We prove Theorem 1-1 in §§1 and 2. The argument in §1 is similar to the argument in Cheeger [2] or Peters [15]. In §2, we treat the ends of  $N$ , making use of a type of  $h$ -cobordism theorem. To deduce Theorem I from Theorem 1-1 we need a description of the topological type of the set

$$\{x \mid \text{the injectivity radius of } N \text{ at } x \text{ is smaller than } a\}.$$

(Here  $a$  is a positive small number.) Theorem 3-1 gives such a description. To prove Theorem 3-1, we use results in Margulis [13] and Gromov [5]. We prove Theorem I in §4, making use of Theorems 1-1 and 3-1.

In §6 we give an estimate on the number of homotopy types containing manifolds satisfying the conditions of Theorem I (Theorem 6-6), making use of Theorem 6-1, which gives an estimate on the number of homotopy types containing manifolds satisfying the conditions of Theorem 1-1. When applied to locally symmetric spaces, Theorems 6-6 together with Mostow's rigidity theorem ([14], [16]) implies the following theorem.

**Theorem II.** *For each positive integer  $n$  greater than 3, there exists a positive number  $C_1$  depending only on  $n$  such that the following holds:*

*Let  $V$  be a positive number. Consider the  $n$ -dimensional rank one locally symmetric spaces whose volumes are smaller than  $V$ . Then, there exist at most  $\exp(\exp(C_1 V))$  isometry classes containing such spaces.*

**Remark.** Gromov, in [7], stated a similar estimate in the case when  $M$  has constant negative curvature. His upper bound is "something like  $V \cdot \exp(\exp(\exp(V + n)))$ ". The author does not know whether his method is similar to ours or not.

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**Notation.**  $D^n(r)$  = the ball of radius  $r$  centered at 0 in the flat  $n$ -dimensional Euclidean space.

For a Riemannian manifold  $N$ , we put

$c^+(N)$  = the supremum of the sectional curvature of  $N$ ,

$c^-(N)$  = the infimum of the sectional curvature of  $N$ ,

$\text{Vol}(N)$  = the volume of  $N$ ,

$\text{diam}(N)$  = the diameter of  $N$ .

For points  $p$  and  $q$  of  $N$  and for a positive number  $r$ , set

$D_p(r) = \{s \in N \mid d(p, s) \leq r\}$ ,

$i_N(p)$  = the injectivity radius of  $N$  at  $p$ , that is  $\sup\{r \mid \text{the restriction to } D^n(r) \text{ of the exponential map } \exp_p: T_p(N) \rightarrow N, \text{ is injective}\}$ ,

$N(r) = \{p \in N \mid i_N(p) \leq r\}$ .

$C_i(\dots)$  and  $\varepsilon_i(\dots)$  always denote the positive constants depending only on the numbers in the parentheses and on the dimension  $n$ .

### 1. A generalization of Cheeger's finiteness theorem—I

We prove Theorem 1-1 in this and the next sections. Since the proof is long, we first give a rough summary of it. In this section we divide the set of all manifolds satisfying the conditions of Theorem 1-1 into finitely many classes, and we construct a local diffeomorphism  $f: N_1 - N_1(a/4) \rightarrow N_2$  for two manifolds  $N_1, N_2$  belonging to the same class. In §2 we modify  $f$  outside the set  $N_1 - \Phi^{-1}(\partial N_1' \times (1/2, 1))$  and make it a PL-homeomorphism between  $N_1$  and  $N_2$ . We need some conditions on  $f$  in order to make the argument in §2 go well. These conditions are listed in Lemma 1-2 below.

Now we start the proof of Theorem 1-1. We may assume that  $N$  is connected, since the number of connected components can be estimated in terms of  $c$  and  $V$ . Set  $\alpha = \min(a/125, (c-d)/6, 1/10)$ , and  $N'' = N' \cup \Phi^{-1}(\partial N' \times (0, 1/2))$ . For a positive number  $d$  smaller than  $c$ , we denote by  $N^{(d)}$  the connected component of  $N - N(d)$  which intersects with  $N'$ .

**Lemma 1-2.** *The set of manifolds satisfying the conditions of Theorem 1-1 is divided into finitely many classes  $\mathcal{C}_1, \dots, \mathcal{C}_\Sigma$  such that the following conditions are satisfied:*

(1)' *For each  $\mathcal{C}_k$  there exists a finite set  $Y_k$ , and for each  $N \in \mathcal{C}_k$  there exists a map  $\Psi_N: Y_k \rightarrow N^{(a/4)}$  such that the  $a$ -neighborhood of  $\Psi_N(Y_k)$  contains  $N^{(a/4)}$ .*

(2') For two manifolds  $N_1, N_2$  belonging to the same class  $\mathcal{C}_k$ , there exists a  $C^\infty$ -map  $f: N_1^{(a/4)} \rightarrow N_2$  which satisfies the following conditions:

(i) For any  $p \in N_1^{(a/4)}$ , there is a neighborhood  $U$  of  $p$  in  $N_1^{(a/4)}$  such that the restriction of  $f$  to  $U$  is a diffeomorphism to its image. (Hereafter we call a map a local diffeomorphism if this condition is satisfied.)

(ii) For each  $p, q \in N_1^{(a/4)}$ , we have  $d(f(p), f(q)) \leq d(p, q) + \alpha$ .

(iii) For each  $i \in Y_k$ , we have  $d(f\Psi_{N_1}(i), \Psi_{N_2}(i)) \leq \alpha$ .

(iv) For each  $i \in Y_k$ , the inequality  $i_{N_1}\Psi_{N_1}(i) \geq 3a/5$  holds if and only if  $i_{N_2}\Psi_{N_2}(i) \geq 3a/5$ .

(v) For each  $i \in Y_k$ , the inequality  $i_{N_1}\Psi_{N_1}(i) \leq (b + c)/2$  holds if and only if  $i_{N_2}\Psi_{N_2}(i) \leq (b + c)/2$ .

*Proof.* The proof of Lemma 1-2 is divided into three steps. In Step 1 we construct the map  $\Psi_N$ . In Step 2 we divide the set of manifolds satisfying the conditions in Theorem 1-1 into finitely many classes  $\mathcal{C}_1, \dots, \mathcal{C}_2$ . In Step 3 we construct the map  $f$ . The method used to construct  $f$  is similar to the argument in Peters [15].

Step 1.

**Assertion 1-3.** There exists a positive number  $C_2$  such that

$$\text{diam}(N^{(a/4)}) \leq C_2 V \alpha^{1-n}$$

holds for every manifold  $N$  satisfying the conditions of Theorem 1-1.

*Proof of Assertion 1-3.* Let  $Z_N = \{p_1, \dots, p_i, \dots\}$  be a maximal subset of  $N^{(a/4)}$  such that  $d(p, q) \geq \alpha/20$  holds for every  $p, q \in Z_N$  with  $p \neq q$ . Then  $D_{p_i}(\alpha/20)$  ( $p_i \in Z_N$ ) are disjoint to each other. On the other hand, since  $i_{N_1}(p_i) \geq \alpha/20$  and  $1 \geq c^+(N)$ , there exists a positive constant  $\delta$  which depends only on  $n$  and which satisfies  $\text{Vol}(D_{p_i}(\alpha/20)) \geq \delta \alpha^n$ .

The above two facts imply

$$(1-1) \quad \#Z_N \leq \text{Vol}(N) / (\delta \alpha^n) \leq FV / (\delta \alpha^n).$$

Now let  $p$  and  $q$  be elements of  $N^{(a/4)}$ . Since  $Z_N$  is maximal, the  $\alpha/10$ -neighborhood of  $Z_N$  contains  $N^{(a/4)}$ . Hence there exist elements  $p_{i(1)}, p_{i(2)}, \dots, p_{i(m)}$  of  $Z_N$  such that the following conditions hold:

(a)  $D_{p_{i(j)}}(\alpha/10) \cap D_{p_{i(j+1)}}(\alpha/10) \neq \emptyset$ .

(b)  $p \in D_{p_{i(1)}}(\alpha/10)$  and  $q \in D_{p_{i(m)}}(\alpha/10)$ .

(c)  $p_{i(j)} \neq p_{i(j')}$  if  $j \neq j'$ .

Condition (c) implies that

$$(1-2) \quad m \leq \#Z_N.$$

For each  $j$  ( $\leq m$ ), fix an element  $q_j$  of  $D_{p_{i(j)}}(\alpha/10) \cap D_{p_{i(j+1)}}(\alpha/10)$ . Then we have

$$(1-3) \quad d(q_j, q_{j+1}) \leq \text{diam}(D_{p_{i(j+1)}}(\alpha/10)) \leq \alpha/5.$$

On the other hand, condition (b) implies

$$(1-4) \quad d(q_1, p) \leq \alpha/10 \quad \text{and} \quad d(q_m, q) \leq \alpha/10.$$

By equations (1-1)–(1-4), we obtain

$$\begin{aligned} d(p, q) &\leq d(p, q_1) + \sum_{j=1}^{m-1} d(q_j, q_{j+1}) + d(q_m, q) \\ &\leq \{2V/(\delta\alpha^n) + 2\} \cdot \alpha/10, \end{aligned}$$

as desired. The proof of Assertion 1-3 is completed.

We construct the map  $\Psi_N$  in the following assertion.

**Assertion 1-4.** *The set of manifolds satisfying the conditions in Theorem 1-1 is divided into finitely many classes  $\mathcal{C}_1^{(1)}, \dots, \mathcal{C}_\Sigma^{(1)}$  such that the following holds:*

(1)'' *For each  $\mathcal{C}_k^{(1)}$  there exists a finite set  $\gamma_k^*$ , and for each  $N \in \mathcal{C}_k^{(1)}$  there exists a map  $\Psi_N: \gamma_k^* \rightarrow N^{(\alpha/4)}$  such that the  $\alpha/10$  neighborhood of  $\Psi_N(Y_k)$  contains  $N^{(\alpha/4)}$ .*

(2)'' *For two manifolds  $N_1, N_2$  belonging to  $\mathcal{C}_k^{(1)}$  and for two elements  $i, j$  of  $Y_k^*$ , the following holds:*

(i)  $d(\Psi_{N_1}(i), \Psi_{N_1}(j)) \leq d(\Psi_{N_2}(i), \Psi_{N_2}(j)) + \alpha/10$ .

(ii)  $i_{N_1}(\Psi_{N_1}(i)) \geq 3\alpha/5$  holds if and only if  $i_{N_2}(\Psi_{N_2}(i)) \geq 3\alpha/5$ .

(iii)  $i_{N_1}(\Psi_{N_1}(i)) \leq (b + c)/2$  holds if and only if  $i_{N_2}(\Psi_{N_2}(i)) \leq (b + c)/2$ .

**Remark.** Conditions (1)'' and (2)'' correspond to the part of conditions (1)' and (2)' in Lemma 1-2 concerning to the image of  $\Psi_N$ .

*Proof of Assertion 1-4.* For a positive number  $m$ , let  $\dot{m}$  denote the set of all positive integers smaller than  $m + 1$ . For each manifold  $N$  satisfying the conditions of Theorem 1-1, fix a subset  $Z_N$  used in the proof of Assertion 1-3, and fix a bijection  $\Psi_N: \#Z_N \rightarrow Z_N$ . Let  $\mathcal{C}_k^{(2)}$  be the set of all manifolds  $N$  which satisfy the conditions of Theorem 1-1 and  $\#Z_N = k$ . Equation (1-1) implies that the sets  $\mathcal{C}_i^{(2)}$  ( $i = 1, 2, \dots$ ) are empty except finitely many ones.

Now (1)'' holds because  $Z_N$  is maximal.

On the other hand, Assertion 1-3 implies that the set  $\{d(\Psi_N(i), \Psi_N(j)) | i, j \leq k, N \in \mathcal{C}_k^{(2)}\}$  is bounded. It follows that we can subdivide the classes  $\mathcal{C}_i^{(2)}$  ( $i = 1, 2, \dots$ ) into finitely many classes  $\mathcal{C}_1^{(1)}, \mathcal{C}_2^{(1)}, \dots, \mathcal{C}_k^{(1)}, \dots$  satisfying (2)''. The proof of Assertion 1-4 is completed.

*Step 2.* In this step, we divide each of the classes  $\mathcal{C}_k^{(1)}$  given in Assertion 1-4 into finitely many classes. Also, we do not change the set  $Y_k$  and the map  $\Psi_N$ .

For  $i \in Y_k$  and  $N \in \mathcal{C}_k^{(1)}$ , we denote by  $\varphi_{N,i}$  the composition of the exponential map,  $\exp_p: T_{\Psi_N(i)}(N) \rightarrow N$  and an origin preserving isometric embedding of  $D^n(6\alpha/10)$  into  $T_{\Psi_N(i)}(N)$ . If

$$(1-5) \quad i, j, N \quad D_{\Psi_N(i)}(2\alpha/10) \cap D_{\Psi_N(j)}(2\alpha/10) \neq \emptyset,$$

then

$$D_{\Psi_N(i)}(2\alpha/10) \subset D_{\Psi_N(j)}(6\alpha/10)$$

and the map  $\varphi_{N_2,j}^{-1}\varphi_{N_1,i}$ :  $D^n(2\alpha/10) \rightarrow D^n(6\alpha/10)$  is well defined. We put  $g_{N;i,j} = \varphi_{N_2,j}^{-1}\varphi_{N_1,i}$ , when (1-5)<sub>i,j,N</sub> holds.

Since  $Y_k$  is a finite set, we can subdivide the classes  $\mathcal{C}_k^{(1)}$  ( $k = 1, 2, \dots$ ) into finitely many classes  $\mathcal{C}_1^{(3)}, \mathcal{C}_2^{(3)}, \dots$  such that the following holds: if  $N_1$  and  $N_2$  are contained in the same class  $\mathcal{C}_k^{(3)}$ , then, for each elements  $i, j$  of  $Y_k$ , (1-5)<sub>i,j,N\_1</sub> holds if and only if (1-5)<sub>i,j,N\_2</sub> holds, in other words  $g_{N_1;i,j}$  is well defined if and only if  $g_{N_2;i,j}$  is well defined.

On the other hand, since the sectional curvatures of our manifolds are uniformly bounded, the maps  $g_{N;i,j}$  are equicontinuous when  $N$  moves in  $\mathcal{C}_k^{(3)}$ . (See [2, Lemma 3.4]).

Therefore, using Ascoli-Arzelà's theorem, we obtain the following. For an arbitrary number  $\theta$  we can subdivide the classes  $\mathcal{C}_k^{(3)}$  ( $k = 1, 2, \dots$ ) into finitely many classes  $\mathcal{C}_1^{(4)}, \mathcal{C}_2^{(4)}, \dots$  such that

$$d(g_{N_1;i,j}(p), g_{N_2;i,j}(p)) \leq \theta$$

holds for each  $p \in D^n(2\alpha/10)$  and  $i, j \in Y_k$ , if  $N_1$  and  $N_2$  are contained in the same class  $\mathcal{C}_k^{(4)}$  and if  $g_{N_1;i,j}$  is well defined. We will fix the number  $\theta$  later.

Suppose  $i$  and  $j$  are elements of  $Y_k$  such that  $g_{N;i,j}$  is well defined. Then  $d(\Psi_N(i), \Psi_N(j)) \leq i_N(\Psi_N(i))$  holds for every  $N \in \mathcal{C}_k^{(4)}$ . Therefore there exists a unique geodesic segment joining  $\Psi_N(i)$  with  $\Psi_N(j)$ . Let  $P_{N;i,j}: T_{\Psi_N(i)}(N) \rightarrow T_{\Psi_N(j)}(N)$  be the parallel displacement along this geodesic.

Since the maps

$$(\Phi_{N,j})_*^{-1}P_{N;i,j}(\Phi_{N,i})_*: T_0(D^n(2\alpha/10)) \rightarrow T_0(D^n(2\alpha/10))$$

are elements of  $SO(n)$ , that is the group of linear isometries of  $T_0(D^n(2\alpha/10))$ , and since  $SO(n)$  is compact, we can subdivide the classes  $\mathcal{C}_k^{(4)}$  ( $k = 1, 2, \dots$ ) into finitely many classes  $\mathcal{C}_1, \mathcal{C}_2, \dots$  such that

$$d\left(\left(\Phi_{N_1,j}\right)^{-1}P_{N_1;i,j}\Phi_{N_1,i}, \left(\Phi_{N_2,j}\right)^{-1}P_{N_2;i,j}\Phi_{N_2,i}\right) \leq \theta$$

holds for each  $N_1$  and  $N_2$  belonging to the same class  $\mathcal{C}_k$ . Here  $d$  is a distance function on  $SO(n)$ .

*Step 3.* We prove that the classes  $\mathcal{C}_k$  divided in Step 2 satisfy conditions (1)', (2)' and (3)' in Lemma 1-2. We have already constructed the set  $Y_k$  and the map  $\Psi_N$  satisfying conditions (1)' and (2)'. We will construct the map  $f$  for each pair  $(N_1, N_2)$  of elements of  $\mathcal{C}_k$ .

Following Peters, we use center of mass technique here. Let  $\eta_1: N_1^{(a/4)} \rightarrow [0, 1]$  be  $C^\infty$ -functions such that  $\sum_{i \in Y_k} \eta_i = 1$  and that the support of  $\eta_i$  is contained in  $D_{\Psi_N(i)}(6\alpha/10)$ .

For  $p \in N_1^{(a/4)}$  and  $q \in N_2$ , set

$$\omega(p, q) = \sum_{i \in Y_k} \left\{ \eta_i(p) \times d\left(\Phi_{N_2, i} \Phi_{N_1, i}^{-1}(p), q\right) \right\}.$$

(Remark that if  $p$  is not contained in the image of  $\Phi_{N_1, i}$ , then  $\eta_i(p) = 0$ . Hence the above function is well defined.)

Let  $f(p)$  be the point of  $N_2$  such that

$$d(p, f(p)) = \min_{q \in N_2} \omega(p, q).$$

The unique existence of such a point  $f(p)$  is known and can be proved by making use of the convexity of the function  $q \rightarrow \omega(p, q)$ . For these facts, see Buser & Karcher [1].

On the other hand, Peters showed that if  $\theta \leq \min(6^{2-n}, \beta/700)$ , then  $f$  is a local diffeomorphism ([15, Lemma], where he treated the case when  $N$  was compact, but his proof can be applied without any change to our case). Put  $\theta = \min(6^{2-n}, \beta/700)$ . Then  $f$  satisfies condition (2)'(i).

By the definition of  $f$ , the inequality

$$d(f(p), \Psi_{N_2}(i)) \leq 6\alpha/10$$

holds if

$$d(p, \Psi_{N_1}(i)) \leq 2\alpha/10.$$

By using this fact and Assertion 1-3, we can easily prove that  $f$  satisfies conditions (2)'(ii), (iii) and (3)'. Thus Lemma 1-2 is proved.

### 2. A generalization of Cheeger's finiteness theorem—II

In this section we complete the proof of Theorem 1-1. Take one of the classes  $\mathcal{C}_k$  given in Lemma 1-2. Suppose  $N_1$  and  $N_2$  belong to  $\mathcal{C}_k$ . If the dimension is not 4, there exist only a finite number of diffeomorphism classes in a given PL-homeomorphism class. Therefore it suffices to show that  $N_1$  is PL-homeomorphic to  $N_2$ . (In the case when  $n = 3$  or 4, it suffices to show that  $N_1$  is homotopy equivalent to  $N_2$ .)

Let  $f: N_1^{(a/4)} \rightarrow N_2$  and  $f': N_2^{(a/4)} \rightarrow N_1$  be the maps given in Lemma 1-2(2).

**Assertion 2-1.** (1)  $f(N_1^{(3a/4)}) \subset N_2^{(a/2)}$ . In particular,  $f'|_{N_1^{(3a/4)}}$  is well defined.



(2) For every point  $p$  of  $N_1^{(3a/4)}$ , we have  $d(f'f(p), p) \leq 7\alpha$ .

*Proof.* Using the fact  $|\text{sectional curvature}| \leq 1$ , we see easily that

$$(2-1) \quad |i_N(p) - i_N(q)| \leq d(p, q)$$

holds for each  $p, q \in N$  with  $d(p, q) \leq \pi/4$ .

Now let  $p$  be an arbitrary point of  $N_1^{(3a/4)}$ . By Lemma 1-2, there exists an element  $j$  of  $Y_k$  such that  $d(p, \Psi_{N_1}(j)) \leq \alpha$ . Therefore, (2-1) implies that

$$i_{N_1}(\Psi_{N_1}(j)) \geq i_{N_1}(p) - \alpha \geq 3a/5.$$

It follows that

$$(2-2) \quad \begin{aligned} i_{N_2}(f(\Psi_{N_1}(j))) &\geq i_{N_2}(\Psi_{N_2}(j)) - d(f(\Psi_{N_1}(j)), \Psi_{N_2}(j)) \\ &\geq 3a/5 - \alpha \geq 11a/20. \end{aligned}$$

The first inequality follows from (2-1). The second inequality follows from Lemma 1-2(iii), (iv) and the fact  $i_{N_1}(\Psi_{N_1}(j)) \geq 3a/5$ . The third inequality follows from the fact  $\alpha \leq a/20$ .

Equation (2-1), together with Lemma 1-2(2)(ii) and the fact  $d(p, \Psi_{N_1}(j)) \leq \alpha$ , imply that

$$i_{N_2}(f(p)) \geq 11a/20 - d(f(p), f(\Psi_{N_1}(q))) \geq 11a/20 - 2\alpha \geq a/2.$$

This proves (1) of Assertion 2-1.

By Lemma 1-2(2)(ii), (iii), we have

$$d(f'f(\Psi_{N_1}(j)), \Psi_{N_1}(j)) \leq d(f'(\Psi_{N_2}(j)), \Psi_{N_1}(j)) + 2\alpha \leq 3\alpha.$$

Using Lemma 1-2(2)(ii), (iii) again, we obtain

$$d(f'f(p), p) \leq d(f'f(\Psi_{N_1}(j)), f'f(p)) + d(\Psi_{N_1}(j), p) + 3\alpha \leq 7\alpha.$$

The proof of Assertion 2-1 is completed.

**Assertion 2-2.** *The restriction of  $f$  to  $N_1'$  is a homotopy equivalence between  $N_1'$  and  $N_2$ .*

*Proof.* Condition (4) in Theorem 1-1 implies that  $N_1'$  is a deformation retract of  $N_1$ . Let  $j_1: N_1 \rightarrow N_1'$  and  $j_2: N_2 \rightarrow N_2'$  be retractions. Set

$$\bar{f} = j_2 f|_{N_1'}, \quad \bar{f}' = j_1 f'|_{N_2'}$$

We will show that  $\bar{f}'\bar{f}$  is homotopic to the identity map of  $N_2'$ . It can be shown in a similar way that  $\bar{f}\bar{f}'$  is homotopic to the identity map of  $N_1'$ . Since the maps  $j_1$  and  $j_2$  are homotopy equivalences, Assertion 2-2 follows from the above two facts.

To prove that  $\bar{f}'\bar{f}$  is homotopic to the identity map, it suffices to show that  $f'f|_{N_1'}$  is homotopic to the inclusion map  $i$  of  $N_1'$  into  $N_1$  (because  $j_1$  and  $j_2$  are homotopy equivalences).

By Assertion 2-1, we have

$$d(f'f(p), p) \leq 7\alpha \leq i_{N_1}(p).$$

Therefore there exists uniquely a minimum geodesic segment  $l: [0, 1] \rightarrow N_1$  joining  $p$  with  $f'f(p)$ . For each element  $t$  of  $[0, 1]$  and for each  $p \in N_2$ , let  $g_t(p)$  denote the point on  $l$  such that  $d(p, g_t(p)) = t \cdot d(p, q)$ . Clearly  $g$  is continuous,  $g_0 = i$  and  $g_1 = f'f$ . Namely  $g$  is a homotopy from  $i$  to  $f'f$ . The proof of Assertion 2-2 is completed.

**Assertion 2-3.** *The restriction of  $f$  to  $N_1''$  is injective.*

*Proof.* Let  $p, q$  be points of  $N_1''$  such that  $f(p) = f(q)$ . (Recall that  $N_1'' = N_1' \cup \Phi^{-1}(\partial N_1' \times [0, 1/2])$ .) We will prove  $p = q$ .

By Assertion 2-1(2), we have

$$d(p, q) \leq d(p, f'f(p)) + d(f'f(q), q) \leq 14\alpha.$$

On the other hand,  $i_{N_1}(p)$  is greater than  $14\alpha$ .

Hence there exists a geodesic  $l: [0, 1] \rightarrow N_1$  such that  $l$  is parametrized proportionally to arc length and that  $l(0) = p$ ,  $l(1) = q$  and that the length of  $l$  is smaller than  $14\alpha$ .

By Lemma 1-2(ii), we have, for each  $s \in [0, 1]$ ,

$$d(f(p), f(l(s))) \leq d(p, l(s)) + \alpha \leq 15\alpha.$$

It follows that

$$fl([0, 1]) \subset D_{f(p)}(15\alpha).$$

Therefore, since  $i_{N_2}(f(p)) \geq 15\alpha$ , there exists a continuous map  $g: [0, 1] \times [0, 1] \rightarrow D_{f(p)}(15\alpha)$  such that  $g(t, 0) = fl(t)$  and  $g(0, t) = g(1, t) = g(t, 1) = f(p)$ .

**Claim.** *There exists a continuous map  $\bar{g}: [0, 1] \times [0, 1] \rightarrow N_1^{(a/2)}$  such that  $f\bar{g} = g$  and  $\bar{g}(t, 0) = l(t)$ .*

Before proving the claim, we prove Assertion 2-4 making use of the claim.

Since  $f\bar{g}(t, 1)$  does not depend on  $t$  and since  $f$  is a local diffeomorphism, it follows that  $\bar{g}(t, 1)$  does not depend on  $t$ . In particular  $\bar{g}(0, 1) = \bar{g}(1, 1)$ . Similarly we obtain  $\bar{g}(1, 0) = \bar{g}(1, 1)$  and  $\bar{g}(0, 1) = \bar{g}(0, 0)$ .

Therefore

$$p = l(0) = \bar{g}(0, 0) = \bar{g}(1, 0) = l(1) = q,$$

as required.

*Proof of the claim.* We will prove by contradiction. Suppose the claim is false. Then the following is valid, since  $f$  is a local diffeomorphism.

There exist  $s_0, t_0 \in ]0, 1[$  and a continuous map  $\bar{g}(t_0, \cdot)$  which maps  $s \in [0, s_0)$  to  $\bar{g}(t, s) \in N_1^{(a/2)}$  and which satisfies  $f(\bar{g}(t_0, s)) = g(t_0, s)$  and  $\bar{g}(t_0, 0) = l(t_0)$ . Furthermore  $s_0$  is the maximum among the numbers which have the above property.

We will deduce a contradiction from (\*). For each  $s \in [0, s_0)$ , we have

$$d(p, \bar{g}(t_0, s)) < d(p, f'f(p)) + d(f'f(p), f'g(t_0, s)) + d(f'f(\bar{g}(t_0, s)), \bar{g}(t_0, s)).$$

This formula, together with Assertion 2-1(2), Lemma 1-2(2)(ii) and the fact  $g(t_0, s) = D_{f(p)}(15\alpha)$ , implies that

$$(2-3) \quad d(p, \bar{g}(t_0, s)) \leq 7\alpha + (d(f(p), g(t_0, s)) + \alpha) + 7\alpha \leq 30\alpha.$$

It follows from (2-3), (2-1) and the fact  $i_{N_1}(p) \geq 3a/5$ , that

$$(2-4) \quad i_{N_1}(g(t_0, s)) \geq 3a/5 - 30\alpha > a/2.$$

Therefore, "all accumulation points of  $\lim_{s \rightarrow s_0} \bar{g}(t_0, s)$  are contained in  $N_1^{(a/2)}$ ".

On the other hand,

$$(2-5) \quad "f \text{ is defined on } N_1^{(a/2)}",$$

$$(2-6) \quad "f \text{ is a local diffeomorphism",}$$

$$(2-7) \quad " \lim_{s \rightarrow s_0} fg(t_0, s) \text{ converges".}$$

The facts (2-4)–(2-7) imply that  $\lim_{s \rightarrow s_0} g(t_0, s)$  converges to a point of  $N_1^{(a/2)}$ . Hence  $g(t_0, \cdot)$  can be extended to  $[0, s_0 + \delta)$  for sufficiently small  $\delta$ . This contradicts (\*). Thus the proof of the claim is completed.

**Assertion 2-4.**  $f(\partial N_1'') \subset N_2 - N_2'$ .

**(Remark.** Condition (4)(ii) in Theorem 1-1 is added to make this assertion valid.)

*Proof of Assertion 2.4.* Let  $p$  be a point of  $\partial N_1''$ . By Lemma 1-2(1)', there exists an element  $j$  of  $Y_k$  such that

$$(2-8) \quad d(p, \Psi_{N_1}(j)) \leq \alpha.$$

Since  $p \in \partial N_1''$ , condition (4)(ii) in Theorem 1-1 implies

$$(2-9) \quad i_{N_1}(p) \leq b.$$

By (2-8), (2-9) and (2-1), we have

$$i_{N_1} \Psi_{N_1}(j) \leq b + \alpha \leq (b + c)/2.$$

Therefore, using Lemma 1-2(2)', we have

$$(2-10) \quad i_{N_2} \Psi_{N_2}(j) \leq (b + c)/2.$$

On the other hand, by Lemma 1-2(2)(ii), we have

$$(2-11) \quad d(f(p), \Psi_{N_2}(j)) \leq d(f(p), f\Psi_{N_1}(j)) + d(f\Psi_{N_1}(j), \Psi_{N_2}(j)) \leq 3\alpha.$$

Inequalities (2-10), (2-11) and (2-1) imply

$$i_{N_2}(f(p)) \leq (b + c)/2 + 3\alpha \leq c.$$

This inequality, together with condition (4)(ii), implies

$$f(p) \in N_2(c) \subset N_2 - N_2',$$

as required.

**Assertion 2-5.** *The restriction of  $f$  to  $\partial N_1''$  is a homotopy equivalence between  $\partial N_1''$  and  $N_2 - N_2'$ .*

The proof of Assertion 2-5 is similar to the proof of Assertion 2-2, and hence is omitted.

Set  $U = \overline{f(N_1'') - N_2'}$  and  $V = \overline{N_2 - f(N_1'')}$ .

**Assertion 2-6.** *The embedding  $i_1$  of  $f(\partial N_1'')$  into  $U$  and the embedding  $i_2$  of  $f(\partial N_1'')$  into  $V$  are homotopy equivalences.*

*Proof.* In order to avoid complicated notations, we assume that  $N_2 - N_2'$  is connected.

By Van-Kampen's theorem, we have

$$\pi_1(N_2 - N_2') \simeq \pi_1(U) \underset{\pi_1(f(\partial N_1''))}{\pi_1} * \pi_1(V).$$

Assertion 2-5 implies that the inclusion map:  $f(\partial N_1'') \rightarrow N_2$  induces an isomorphism on fundamental groups.

Using these facts, we can prove easily that  $i_1$  and  $i_2$  induce isomorphisms on fundamental groups.

On the other hand, by using Assertion 2-6 and the Mayer-Vietoris exact sequence

$$\dots \rightarrow H_*(f(\partial N_1'')) \rightarrow H_*(U) \oplus H_*(V) \rightarrow H_*(N_2 - N_2') \rightarrow \dots$$

we can prove easily that  $i_1$  and  $i_2$  induce isomorphisms on homotopy groups of any local coefficient system.

Assertion 2-6 follows easily from these two facts.

**Assertion 2-7.** *If  $n \neq 3, 4$ , then  $V$  is PL-homeomorphic to  $f(\partial N_1'') \times [0, 1)$ .*

*Proof.* By condition (4) in Theorem 1-1, we can attach to  $N_2$  a boundary  $\partial \bar{N}_2$ , which is PL-homeomorphic to  $\partial N_2''$ , and can make  $N_2$  a compact PL-manifold  $\bar{N}_2$ . Set  $\bar{V} = V \cup \partial \bar{N}_2$ .

Since the restriction of  $f$  to  $N_1'$  is a PL-homeomorphism to its image and since  $\partial N_1''$  is contained in  $N_1'$ , it follows that  $f(N_1'')$  is a PL-submanifold of  $\bar{N}_2$ .

Now Assertion 2-7 implies that  $(\bar{V}, f(\partial N_1''))$  and  $(\bar{V}, \partial \bar{N}_2)$  are  $\infty$ -connected. Therefore, Theorem 7-11 in [13] implies that  $\bar{V} - \partial \bar{N}_2 (= V)$  is PL-homeomorphic to  $f(\partial N_1'') \times [0, 1)$ , as required.

Assertion 2-7 implies that  $f|_{N_1''}$  is extended to a PL-homeomorphism from  $N_1$  to  $N_2$ . This completes the proof.

### 3. Negatively curved manifolds

In this section we review negatively curved manifolds. First we need some notations. Let  $M$  be a complete Riemannian nonpositively curved manifold. Let  $X$  be the universal covering space of  $M$ , and let  $\pi$  be the natural projection  $\pi: X \rightarrow M$ , and let  $\Gamma$  be the group  $\pi_1(M)$  acting on  $X$  as the group of covering transformations.

In this section and the next, we assume either that  $X$  satisfies the visibility axiom of Eberlein & O'Neill [3] or that  $M$  has negative curvature (namely, for each point  $p$  of  $X$  and for each plane  $\pi \subset T_p(X)$ , the sectional curvature of  $X$  at  $\pi$  is strictly negative).

Let  $\gamma$  be a selfisometry of  $X$ . We call  $\gamma$  an elliptic isometry if  $\gamma$  has a fixed point in  $X$ , a hyperbolic isometry if  $\gamma$  has a unique invariant geodesic and has no fixed point in  $X$ , and a parabolic isometry if  $\gamma$  is neither elliptic nor hyperbolic. For two points  $p$  and  $q$  of  $X$ , we denote by  $\overline{pq}$  the geodesic joining  $p$  with  $q$ .

For a point  $p$  of  $X$  and subset  $A$  of  $\Gamma$ , we set  $\delta_A(p) = \inf_{\gamma \in A - \{1\}} d(p, \gamma(p))$ . If  $A$  is invariant under the inner automorphisms of  $\Gamma$ , then  $\delta_A$  is invariant by the action of  $\Gamma$ . Hence  $\delta_A$  induces a function on  $M (= \Gamma \backslash X)$ . We denote this function also by  $\delta_A$ . It is easy to see that

$$\delta_\Gamma/4 \leq i_M \leq \delta_\Gamma.$$

We set

$$\begin{aligned} X_{a,A} &= \{ p \in \Gamma | \delta_A(p) \leq a \}, \\ M_{a,A} &= \{ p \in M | \delta_A(p) \leq a \}, \\ X_a &= X_{a,\Gamma}, \quad M_a = M_{a,\Gamma}, \\ \Gamma_{\epsilon,p} &= \text{subgroup of } \Gamma \text{ generated by } \{ \gamma | d(\gamma(p), p) < \epsilon \}. \end{aligned}$$

Now we give a description of the set  $M_a$ . Assume  $M$  satisfies  $c^-(M) \geq -1$ , and  $\text{Vol}(M) < \infty$ . Let  $\epsilon_2$  be the Margulis' constant (see Gromov [5, 3-2] or Buser & Karcher [1, 2-5]). Let  $\epsilon$  be a positive number smaller than  $\epsilon_2$ . We

denote by  $S_1, S_2, \dots$  all connected components of  $M_\varepsilon$ . Choose, for each  $i$ , one of the connected components of  $\pi^{-1}(S_i)$ , and denote it by  $\tilde{S}_i$ . In the case when  $X$  satisfies the visibility axiom, Eberlein [4, Corollary 3-3] implies that there uniquely exists a maximal almost nilpotent subgroup containing  $\Gamma_{\varepsilon,p}$ . (Here we say a group is almost nilpotent if it has a nilpotent subgroup with finite index.) It is easy to see that this group depends only on  $i$  and does not depend on  $p \in \tilde{S}_i$ . We denote this subgroup by  $\Gamma_i$ .

**Theorem 3-1.** *Suppose that  $X$  satisfies the visibility axiom. Then, for each  $i$ , one of the following statements holds:*

- (1) (a)  $S_i$  is diffeomorphic to an  $\mathbf{R}^{n-1}$ -bundle over  $S^1$ .
- (b)  $\Gamma_i$  is isometric to  $\mathbf{Z}$  and all nontrivial elements of  $\Gamma_i$  are hyperbolic.
- (c) There exists a geodesic  $l$  such that  $l$  is invariant by all elements of  $\Gamma_i$  and that  $\pi(l)$  is a closed geodesic contained in  $S_i$ .
- (2) (a)  $\Gamma_i$  acts on  $\mathbf{R}^{n-1}$  freely such that
  - (i)  $\Gamma_i \backslash \mathbf{R}^{n-1}$  is compact.
  - (ii)  $S_i$  is homeomorphic to  $[0, 1) \times (\Gamma_i \backslash \mathbf{R}^{n-1})$ .
- (b) There exists a unique point on  $\partial X$  which is invariant by  $\Gamma_i$ .

In the case when we do not assume the visibility axiom for  $X$  and when we assume that  $X$  has negative curvature, an analogue of Theorem 3-1 holds. But in this case, we do not know whether  $S_i$  is homeomorphic to  $[0, 1) \times \Gamma_i \backslash \mathbf{R}^{n-1}$  in case (2).

**Theorem 3-2.** *Suppose  $X$  has negative curvature. Then, for each  $i$ , one of the following statements holds:*

- (1) For each  $p \in S_i$ , there exists uniquely a maximal almost nilpotent subgroup  $\Gamma_{p,i}$  containing  $\Gamma_{\varepsilon,p}$ . The group  $\Gamma_{p,i}$  does not depend on  $p$  and depends only on  $i$ . Put  $\Gamma_i = \Gamma_{p,i}$ . Then conditions (a), (b) and (c) in Theorem 3-1(1) hold.
- (2) There exist a compact manifold  $L$  and a homeomorphism  $\Phi$  between  $S_i$  and  $L \times [0, 1)$  such that conditions (a) and (b) below hold for each  $p \in L$ .
  - (a) If  $t_1, t_2 \in [0, 1)$  and if  $t_1 < t_2$ , then we have

$$\delta_\Gamma(\Phi^{-1}(p, t_1)) > \delta_\Gamma(\Phi^{-1}(p, t_2)).$$

- (b)  $\lim_{t \rightarrow 1} \delta_\Gamma(\Phi^{-1}(p, t)) = 0$ .

*Proofs of Theorems 3-1 and 3-2.* First we need a lemma.

**Lemma 3-3.** *If  $p_0 \in S_i$  and if  $\Gamma_{\varepsilon,p_0}$  contains a hyperbolic isometry, then, for every  $p \in S_i$ , the group  $\Gamma_{\varepsilon,p}$  contains a hyperbolic isometry.*

*Proof of Lemma 3-3.* Set  $U = \{p \in S_i \mid \Gamma_{\varepsilon,p} \text{ contains a hyperbolic isometry.}\}$

*Claim 1.*  $U$  is open.

*Proof.* Let  $p$  be an element of  $U$ . By the definition of  $U$ , there exists a hyperbolic element in  $\Gamma_{\varepsilon,p}$ . Therefore, Gromov [6, 2-5] implies that all non-trivial elements of  $\Gamma_{\varepsilon,p}$  are hyperbolic. Hence there exists a hyperbolic isometry

$\gamma$  such that  $\delta_{\{\gamma\}}(p) < \epsilon$ . Then there exists a neighborhood  $W$  of  $p$  such that  $\delta_{\{\gamma\}}$  is smaller than  $\epsilon$  in  $W$ . Hence  $W \subset U$ , as required.

*Claim 2.*  $U$  is closed in  $S_i$ .

*Proof.* Let  $p$  be an element of  $S_i \cap \bar{U}$ . Since  $p$  is contained in  $S_i$ , there exists a nontrivial element  $\gamma$  of  $\Gamma$  such that  $\delta_{\{\gamma\}}(p) < \epsilon$ . Hence there exists a neighborhood  $W$  of  $p$  such that  $\delta_{\{\gamma\}}$  is smaller than  $\epsilon$  on  $W$ . Since  $p \in \bar{U}$ , it follows that  $W \cap U \neq \emptyset$ . Let  $q \in W \cap U$ . Since  $\Gamma_{\epsilon,q}$  contains a hyperbolic isometry, all elements of  $\Gamma_{\epsilon,q}$  are hyperbolic [5, 2-5]. Hence  $\gamma$  is hyperbolic. It follows that  $p \in U$ , as desired.

Since  $S_i$  is connected, Lemma 3-3 follows from Claims 1 and 2.

We return to the proofs of Theorems 3-1 and 3-2. We show that (1) holds if  $\Gamma_{\epsilon,p}$  contains a hyperbolic element and (2) holds if  $\Gamma_{\epsilon,p}$  does not contain a hyperbolic element.

*Case 1.* The case when  $\Gamma_p$  contains a hyperbolic element.

In this case, Lemma 3-3 and [5, 2-5] imply that there exists uniquely a maximal almost nilpotent subgroup  $\Gamma_{p,i}$  containing  $\Gamma_{\epsilon,p}$ , and that  $\Gamma_{p,i}$  does not depend on  $p$ . Then, Margulis' lemma ([1, 2-5], [5, 3-2]) and [5, 2-5] imply that  $\Gamma_i \simeq \mathbf{Z}$  and that all nontrivial elements of  $\Gamma_i$  are hyperbolic with the same invariant geodesic  $l$ . It is easy to see that  $\tilde{S}_i$  contains  $l$ . Let  $p$  be an element of  $\tilde{S}_i$  and  $q$  be the element of  $l$  such that  $d(p, q) = d(p, l)$ . Then, it is easy to see that  $\bar{p}q \subset \tilde{S}_i$ . On the other hand, [5, 3-4] implies that  $S_i = \Gamma_i \setminus \tilde{S}_i$ . Therefore  $S_i$  is diffeomorphic to an  $\mathbf{R}^{n-1}$ -bundle over  $S^1$ . Thus we have proved that (1) holds in this case.

*Case 2.* The case when  $\Gamma_{\epsilon,p}$  does not contain a hyperbolic element.

In this case, we must prove Theorems 3-1 and 3-2 separately.

*Proof of Theorem 3-1.* Since all elements of  $\Gamma_i$  are parabolic, Eberlein [4, Corollary 3-3] implies that there exists  $p_0 \in \partial X$  such that  $\Gamma_{p_0} = \Gamma_i$ . Since a parabolic isometry has only one fixed point, it follows that  $p_0$  is uniquely determined.

We need some facts on horosphere here.

Let  $p_0$  be a point on  $\partial X$ . We define the Buseman function  $\beta_{p_0}$  as follows. Take  $q \in X$ , and let  $l: [0, \infty) \rightarrow X$  be the half geodesic satisfying  $l[0, \infty) = \bar{p_0q} - \{p_0\}$ . For an element  $p$  of  $X$ , set  $\beta_{p_0}(p) = \lim_{t \rightarrow \infty} (d(l(t), p) - t)$ . Then, it is proved in Eberlein-O'Neill [3, p. 56, Propositions 3-1 and 3-5], that the above functions converge to a  $C^1$ -function of  $p$ , and the resulting function does not depend on  $q$  modulo a constant number. Let us denote the limit by  $\beta_{p_0}$ . For a point  $p_0$  on  $\partial X$ , a horosphere of  $p_0$  is a set which is  $\{p \in X | \beta_{p_0}(p) = a\}$  for some positive number  $a$ .

**Lemma 3-4.** *Any horosphere is diffeomorphic to Euclidean space.*

*Proof.* Set  $V = \{p \in X \mid \beta_{p_0}(p) = a\}$ . We prove that  $V$  is diffeomorphic to  $\mathbf{R}^{n-1}$ .

Take  $q \in X - V$  such that  $\overline{qp_0} \cap V \neq \emptyset$ . It is easy to see that  $\overline{qp_0} \cap V$  consists of one point. Let  $o$  denote this point. Put  $\bar{x} = \text{grad}(\beta_{p_0})$ . Then we see

$$\bar{x}(p) = \frac{d}{dt}(\overline{qp_0}(t))|_{t=0} \quad (t \in T_p(X)).$$

Hence  $\beta_{p_0}$  has no singular point. It follows that  $V$  is a  $C^1$ -submanifold of  $X$ . Denote by  $\bar{y}$  the vector field on  $V$  such that  $\bar{y}(p)$  is the orthogonal projection to  $T_p(V)$  of  $d/dt\overline{qp}(t)|_{t=0} (\in T_p(X))$ . The vector field  $\bar{y}$  is continuous but not necessarily differentiable.

**Assertion 3-5.** *If  $p \neq o$ , then  $\bar{y}(p) \neq 0$ .*

*Proof.* If  $\bar{y}(p) = 0$ , then  $d/dt\overline{qp}(t)|_{t=0}$  is parallel to  $\bar{x}(p)$ . Hence  $q, p$  and  $p_0$  lie on one geodesic. It follows that  $q \in \overline{qp_0}$ . Therefore  $p = o$ , as desired.

We return to the proof of Lemma 3-4. For a point  $p$  of  $V$ , set  $g(p) = d(p, q)$ . It is easy to see that  $\text{grad}(g) = \bar{y}$ . Hence, Assertion 3-5 implies that  $g$  has only one critical point  $o$  in  $V$ .

If  $o$  is a nondegenerate critical point, Morse theory would imply the lemma. But we do not know this fact. Hence we proceed as follows.

Choose a  $C^\infty$ -structure on  $V$  which is compatible with the  $C^1$ -structure as a submanifold of  $X$ . Let  $D$  be a neighborhood of  $o$  in  $V$  such that  $D$  is diffeomorphic to  $D^{n-1}$ . Let  $U$  and  $U'$  be open subsets of  $V$  such that  $o \in U \subset \bar{U} \subset U' \subset \bar{U}' \subset \text{Int}(D)$ .

Since  $\bar{y}(p)$  is not equal to 0 for each  $p$  not equal to  $o$ , it follows that there exists a  $C^1$ -function  $g'$  on  $V$  such that the following conditions are satisfied:

- (1)  $g'$  is of  $C^\infty$  class on  $V - U$ .
- (2)  $g'$  coincides with  $g$  in a neighborhood of  $o$ .
- (3)  $g'$  is nonsingular on  $V - \{o\}$ .
- (4) For every positive number  $b$ , the set  $\{p \in V \mid g'(b) \leq b\}$  is compact.

Let  $\varphi$  be a  $C^\infty$ -function on  $V$  satisfying the following conditions:

- (1)  $\varphi(p) = 0$ , for  $p \in U'$ .
- (2)  $\varphi(p) = 1$ , for  $p \in X - D$ .
- (3)  $0 \leq \varphi(p) \leq 1$ .

Choose a ( $C^\infty$ -) Riemannian metric on  $V$  and set  $\bar{z} = \varphi \text{grad } g'$ .

By condition (1) on  $g'$  and condition (1) on  $\varphi$ , the vector field  $\bar{z}$  is of  $C^\infty$  class. Hence there exists a one-parameter family of transformations  $\Phi_t$  associated with  $\bar{z}$ .

Condition (4) on  $g'$  and condition (2) on  $\varphi$  imply the following: for each compact subset  $K$  of  $V$  there exists a positive number  $t(K)$  such that  $\Phi_t(D)$  contains  $K$  for each  $t \geq t(K)$ .



Take a sequence of compact sets  $K_1, K_2, \dots$  such that  $K_i \subset K_{i+1}$ , and  $\bigcup_{i=1} K_i = V$ . Define positive numbers  $t_1, t_2, \dots$  inductively as follows:  $t_1 = t(D \cup K_1), \dots, t_{i+1} = t(\Phi_{t_i}(D) \cup K_i)$ . Set  $D_i = \Phi_{t_i}(D)$ . Then  $D_i \subset D_{i+1}$ ,  $\bigcup_{i=1} D_i = V$  and each  $D_i$  is diffeomorphic to  $D^{n-1}(1)$ .

Therefore, the annulus theorem (see for example [12, Corollary 2-16-1]) implies that  $D_i - D_{i-1}$  is PL-homeomorphic to  $S^{n-2} \times [0, 1)$ . Therefore  $V$  is diffeomorphic to  $\mathbf{R}^{n-1}$  (see [11]). The proof of Lemma 3-4 is completed.

We return to the proof of Theorem 3-1.

We have shown that there exists uniquely an element  $p_0$  of  $X$  such that  $\Gamma_i = \Gamma_{p_0}$ . It follows easily that

$$\tilde{S}_i = \{ p \in X \mid \delta_{\Gamma_i}(p) < \varepsilon \}.$$

By a method similar to the proof in [5, 3-4], we can prove that

$$(3-1) \quad S_i \simeq \Gamma_i \setminus \tilde{S}_i.$$

We see easily that

$$(3-2) \quad \overline{p p_0} \cap X \subset \tilde{S}_i \quad \text{for each } p \in \tilde{S}_i.$$

On the other hand, Eberlein [4, Lemma 3-1(e)] implies that there exists a positive number  $a$  such that

$$(3-3) \quad \{ p \in X \mid \beta_{p_0}(p) \leq a \} \subset \tilde{S}_i.$$

Now set  $\text{grad}(\beta_{p_0}) = \bar{x}$ . Since  $\bar{x}$  is invariant by the action of  $\Gamma_i$ , it follows that  $\bar{x}$  induces a vector field on  $S_i = \Gamma_i \setminus \tilde{S}_i$ . We denote this vector field also by  $\bar{x}$ . Take the number  $a$  given in (3-3), and put  $S'_i = \{ p \in M \mid \beta_{p_0}(p) \leq a \}$ .

By (3-2), we obtain a homeomorphism  $h: S_i \rightarrow S'_i$  such that, for  $p \in \tilde{S}_i$ , two points  $p$  and  $h(p)$  are contained in the same orbit of  $\bar{x}$ .

On the other hand, if we let  $V$  denote  $\{ p \in X \mid \beta_{p_0}(p) = a \}$ , then Lemma 3-4 implies that  $V$  is diffeomorphic to  $\mathbf{R}^{n-1}$ . On the other hand,  $S'_i \simeq \Gamma_i \setminus V \times [0, 1)$ . Therefore  $S_i \simeq \Gamma_i \setminus V \times [0, 1)$ .

Since  $\partial S_i$  is closed in  $M - \bigcup_i \text{Int } S_i$ , and since  $M - \bigcup_i \text{Int } S_i$  is compact, it follows that  $\partial S_i \simeq \Gamma_i \setminus V$  is compact. Thus the proof of Theorem 3-1 is completed.

*Proof of Theorem 3-2 in the case when  $\Gamma_{\varepsilon, p}$  contains no hyperbolic elements.*

**Assertion 3-6.** *Suppose  $\Gamma_{\varepsilon, q}$  contains no hyperbolic elements for each  $q \in \tilde{S}_i$ . Then there exists a  $C^\infty$ -vector field  $\bar{x}$  on  $S_i$  such that the following holds:*

*Let  $p$  be an arbitrary point of  $S_i$ , and let  $\gamma_1, \gamma_2, \dots$ , be all elements of  $\Gamma$  such that  $\delta_{\{\gamma_j\}}(p) = \delta_\Gamma(p)$ . Then, for each  $j$ , we have*

$$(3-4) \quad (\bar{x}(p))(\delta_{\{\gamma_j\}}) > 0.$$

*Proof.* Margulis' lemma and [5, 2-7], imply that, for each  $p \in S_i$ , there exists a vector  $\vec{x}(p) \in T_p(X)$  such that (3-4) holds for each  $j$ . Then Assertion 3-6 follows by using a partition of unity.

Now we can complete the proof of Theorem 3-2. Assertion 3-6 immediately implies that  $L = \{p \in S_i | \delta_\Gamma(p) = \varepsilon/2\}$  is a topological submanifold of  $S_i$ . It is easy to see that  $L$  is compact.

Let  $\Psi_t$  be the 1-parameter group of transformations associated to  $\vec{x}$ . For each point  $p$  of  $L$ , the intersection of  $\partial S_i$  and  $\{\Psi_t(p) | t \in \mathbf{R}\}$  consists of one point, which we denote by  $\Psi_{g(t)}(p)$ . For  $(p, t) \in L \times [0, 1)$ , set

$$F(p, t) = \Psi_{g(t) + \tan(\pi t/2)}(p).$$

Then, clearly,  $F: L \times [0, 1) \rightarrow S_i$  is a homeomorphism. Set  $\Phi = F^{-1}$ . Condition (2)(a) follows from Assertion 3-6.

*Proof of condition (2)(b).* If  $\limsup_{t \rightarrow 1} \delta_\Gamma(\Phi^{-1}(p, t)) \neq 0$ , then there exists a sequence of elements,  $t_1, t_2, \dots$  of  $[0, 1)$  and a positive number  $\theta$  such that

$$(3-5) \quad \tan(\pi t_{i+1}/2) - \tan(\pi t_i/2) > 1,$$

$$(3-6) \quad \delta_\Gamma(\Psi^{-1}(p, t_i)) > \theta.$$

On the other hand, since the set  $K = \{q \in S_i | \delta_\Gamma(p) > \theta\}$  is compact, there exists a positive number  $\lambda$  such that

$$(3-7) \quad \delta_\Gamma(q) - \delta_\Gamma(\Phi_t(q)) < -\lambda$$

for every  $q \in K$  and  $t \geq 1$ .

Equations (3-5)–(3-7) imply that

$$\delta_\Gamma(\Phi^{-1}(p, t_i)) < \delta_\Gamma(\Phi^{-1}(p, t_{i-1})) - \lambda.$$

This contradicts (3-6). The proof of Theorem 3-2 is now completed.

We call  $S_i$  an  $\varepsilon$ -tube if (1) is satisfied and we call  $S_i$  an  $\varepsilon$ -cusp if (2) is satisfied.

**Remark 3-7.** For each point  $p$  of  $X$ , there exist a finite number of elements  $\gamma_1, \gamma_2, \dots, \gamma_k$  of  $\Gamma$  such that  $\delta_\Gamma = \min_{j=1}^k (\delta_{\gamma_j})$  on some neighborhood of  $p$ . Using this fact, we see that  $L$  is a PL-submanifold and that  $\Phi$  is a PL-homeomorphism.

#### 4. Proof of Theorem I

In this section, we prove Theorem I. We need the following lemma of Gromov, which played the key role in the proof by Gromov of his finiteness theorem.

In this section, we assume either that  $M$  has negative curvature or that the universal covering space of  $M$  satisfies the visibility axiom.

Let  $n \geq 4$  be an integer. Let  $\epsilon$  be a number smaller than  $\epsilon_2/2$ . Suppose  $M$  satisfies  $\text{Vol}(M) < \infty$ ,  $c^-(M) \geq -1$  and  $\dim(M) = n$ . Let  $S$  be an  $\epsilon$ -tube of  $M$  and  $S'$  be the  $2\epsilon$ -tube containing  $S$ , and let  $\pi(l)$  be the closed geodesic which is contained in  $S$  and whose length is smaller than  $\epsilon$ . Assume  $d(\partial S, \pi(l)) > 2\epsilon$  and  $d(M - S, \pi(l)) > 3$ .

**Lemma B** (Gromov [5, 4.4]). *There exists a positive number  $C_3$  such that*

$$\text{Vol}(S') \geq C_3 \cdot \text{diam}(S)^{p_n} \epsilon^n,$$

where  $p_n = 1$  if  $n \geq 8$ ,  $p_n = 3/2$  if  $n = 6$  or  $7$ ,  $p_n = 3$  if  $n = 4$  or  $5$ .

Gromov, in [5], also remarked the following: for each  $\delta > 0$  we have a constant  $C_4(\delta)$  such that,

$$(4-1) \quad \text{Vol}(S') \geq C_4(\delta) \cdot \text{diam}(S) \cdot \epsilon^n$$

holds for  $M$  satisfying  $c^+(M) < -\delta$  in addition.

Gromov deduced, from Lemma B, the inequality

$$\text{diam}(M) \leq \text{const} \cdot \text{Vol}(M)^{p_n}$$

in the case when  $M$  is compact. This formula does not hold in the case when  $M$  is noncompact. But we can prove a similar formula.

**Theorem 4-1.** *For each positive integer  $n$  with  $n \geq 4$  and for each positive number  $\epsilon$  with  $\epsilon \leq \epsilon_2/2$ , there exists a positive number  $C_5(\epsilon)$  such that the following holds. If  $M$  is connected and satisfies  $\dim(M) = n$  and  $c^-(M) \geq -1$ , then we have*

$$\text{diam}(M - (\text{the union of all } \epsilon\text{-cusps})) \leq C_5(\epsilon) \cdot \text{Vol}(M)^{p_n}.$$

**Theorem 4-1'.** *For each positive number  $\delta$  and  $\epsilon \leq \epsilon_2/2$ , there exists a positive number  $C_6(\epsilon, \delta)$  such that*

$$\text{diam}(M - (\text{the union of all } \epsilon\text{-cusps})) \leq C_6(\epsilon, \delta) \cdot \text{Vol}(M)$$

holds if  $M$  satisfies  $c^+(M) < -\delta$  in addition.

*Proof of Theorems 4-1 and 4-1'.* By  $i_M \leq \delta_\Gamma$ , we have

$$(4-2) \quad M - M_\epsilon \subset M - M(\epsilon).$$

Hence Assertion 1-3 and (4-2) imply

$$\text{diam}(M - M_\epsilon) \leq C_2 \cdot \text{Vol}(M) \cdot \epsilon^{1-n}.$$

On the other hand, Lemma B implies

$$(4-4) \quad \text{diam}(\text{an } \epsilon\text{-cusp of } M) \leq (C_3^{-1} \cdot \text{Vol}(M) \cdot \epsilon^{-n})^{p_n}.$$

Theorem 4-1 immediately follows from (4-3) and (4-4). The proof of Theorem 4-1' is similar.

**Theorem 4-2.** For each integer  $n$  with  $n \geq 4$ , and for each positive number  $V$ , there exists a positive number  $\epsilon_3(V)$  such that the following holds. If an  $n$ -dimensional manifold  $M$  satisfies  $\text{Vol}(M) < V$  and  $c^-(M) \geq -1$ , then  $M$  contains no  $\epsilon_3(V)$ -tubes. In other words,  $M_{\epsilon_3(V)}$  consists only on cusps.

*Proof.* We may assume that  $M$  is connected. Since  $M - M_{\epsilon_2} \neq \emptyset$ , we have  
 (4-5)  $\text{Vol}(M - (\epsilon_2/2\text{-cusps})) \geq \text{Vol}(D^n(1)) \cdot (\epsilon_2/4)^n$ .

Set

$$\epsilon_3(V) = \epsilon_1((\epsilon_2/4)^n \cdot \text{Vol}(D^n(1)), C_5(\epsilon_2/2) \cdot V^{p_n}).$$

We will prove by contradiction that  $M$  has no  $\epsilon_3(V)$ -tubes.

Suppose that  $S$  is an  $\epsilon_3(V)$ -tube of  $M$ . Take a point  $p_0$  on  $l$ . We see

(4-6)  $\text{length of } l \leq \epsilon.$

Since  $p_0$  is contained in a tube, it is not contained in any cusps. On the other hand, Theorem 4-1 implies that

(4-7)  $\text{diam}(M - (\epsilon_2/2\text{-cusps})) \leq C_5(\epsilon_2/2) \cdot V^{p_n}.$

Therefore, Theorem A, the definition of  $\epsilon_3(V)$  and equations (4-6) and (4-7) imply

$$\text{Vol}(M - (\epsilon_2/2\text{-cusps})) < (\epsilon_2/4)^n \cdot \text{Vol}(D^n(1)).$$

This contradicts (4-5). The proof of Theorem 4-2 is completed.

*Proof of Theorem I.* Now we can prove Theorem I. First we assume  $n \geq 4$ . Set  $a = \epsilon_3(V)/64$ ,  $b = \epsilon_3(V)/16$  and  $c = \epsilon_3(V)/8$ . Suppose  $M$  satisfies the conditions of Theorem I. It suffices to show that  $M$  satisfies the conditions of Theorem 1-1 when we take the numbers  $a, b$  and  $c$  as above.

It is clear that  $M$  satisfies Conditions (1), (2) and (3) in Theorem 1-1. We will verify Condition (4). We treat only the case when the universal covering space of  $M$  satisfies the visibility axiom. The proof in the case when  $M$  has negative curvature is similar (use Theorem 3-2 instead of Theorem 3-1).

Set  $M' = M_{\epsilon_3(V)}$ . Theorem 4-2 implies that  $M_{\epsilon_3(V)}$  consists of cusps. Take one of the cusps, and let us denote it by  $S_i$ . Let  $p$  be the point on  $\partial X$  such that  $\Gamma_i = \Gamma_{p_0}$ . (We are using the notation used in Theorem 3-1.) Denote by  $\Phi_i$  the PL-homeomorphism between  $S_i$  and  $[0, 1) \times (\Gamma_i \setminus \mathbf{R}^{n-1})$  given in Theorem 3-2. (As was remarked in 3-7, the homomorphism given in Theorem 3-2 is a PL-homeomorphism.)

Then  $\Phi_i^{-1}([0, 1) \times \{\text{a point}\})$  is an orbit of  $\vec{x}$ . (Here  $\vec{x}$  is the gradient vector field of Buseman function.) Hence there exists a PL-homeomorphism  $\Phi'_i: S_i \rightarrow \partial S_i \times [0, 1)$  such that

$$\Phi_i^{-1}(\partial S_i \times \{1/2\}) = \{p \in S_i \mid \delta_{\Gamma}(p) = \epsilon_3(V)/16\}.$$

Let  $\Phi: M - M' \rightarrow (0, 1) \times \partial M'$  be the PL-homeomorphism satisfying  $\Phi|_{S_i} = \Phi'_i$ .

Now we will show that condition (3) is satisfied if we take this function  $\Phi$  and this open subset  $M'$ . (i) is clear.

*Proof of (ii).* We have

$$M' \cap M(c) = \{ p \in M \mid \delta_{\Gamma}(p) \geq \varepsilon_3(V), i_M(p) \leq \varepsilon_3(V)/8 \}.$$

On the other hand, we see  $\delta_{\Gamma}/4 \leq i_M$ . Therefore we have  $M' \cap M(c) = \emptyset$ , as desired.

*Proof of (iii).* Suppose  $p$  is an element of  $\Phi^{-1}(\partial M' \times \{1/2\})$ . By the definition of  $\Phi$ , we have  $\delta_{\Gamma}(p) = \varepsilon_3(V)/16$ . Hence

$$a = \varepsilon_3(V)/64 \leq i_M(p) \leq \varepsilon_3(V)/16 = b.$$

Therefore we have  $p \in M(b) - M(a)$ , as desired.

Thus we have proved Theorem I in the case when  $n \geq 4$ .

In the case when  $n = 2$ , Theorem I can be easily deduced from Gromov's Betti number estimate  $b_i(M) \leq \text{const} \cdot \text{Vol}(M)$  [8, p. 12].

### 5. A counterexample

In this section, we give an example which shows that we cannot replace Condition (2) in Theorem I by "M has nonpositive curvature".

**Proposition 5-1.** *For each positive integer n greater than 2, there exists a sequence of Riemannian manifolds  $M_1, M_2, \dots$  which has the following properties:*

- (1) *The volumes of  $M_i$  ( $i = 1, 2, \dots$ ) are uniformly bounded.*
- (2) *There exists a positive number  $C_7$  such that*

$$0 \geq \text{sectional curvature of } M_i \geq -C_7$$

*holds for each i.*

- (3) *If  $i \neq j$ , then the Betti numbers of  $M_i$  and  $M_j$  are distinct.*

Since our construction is quite similar to the one in Gromov [5], we give only an outline of the construction. (See also Eberlein [4, p. 459].) First we study the case when  $n = 3$ . In [5], Gromov took infinitely many manifolds  $X_i$ , which are diffeomorphic to (Torus—two open disks)  $\times$  circle. And he made a manifold  $M_\infty$  by identifying one of the boundaries of  $X_i$  to that of  $X_{i-1}$  and by identifying the other boundary of  $X_i$  to that of  $X_{i+1}$ . Then  $M_\infty$  is a nonpositively curved manifold which has finite volume and bounded curvature but whose Betti number is infinite. Now, we take finitely many manifolds  $X_1, \dots, X_k$  from  $X_i$  ( $i = 1, 2, \dots$ ). Then we get a manifold  $M'_k$  by identifying their boundaries in a similar way. The boundary of  $M'_k$  is one of the boundaries of

$X_k$ . We modify  $M'_k$  and obtain a closed manifold  $M_k$ . These manifolds have required properties.

In the case when  $n \neq 3$ , we take  $M_k \times (S^1)^{n-3}$ .

**6. An estimate on the number of homotopy types**

In this section, we give an estimate on the number of homotopy types containing manifolds satisfying the conditions of Theorem 1-1.

**Theorem 6-1.** *In each dimension, there exists a positive number  $C_8$  such that the following holds. For each positive number  $a, b, c$  and  $V$  satisfying  $a < b < c$  and  $V \leq 1$ , there exist at most  $(d^{-n}V)^{C_8 d^{-n}V}$  homotopy types containing manifolds  $N$  satisfying the conditions of Theorem 1-1. Here we put  $d = \min\{a/3, (b - c)/2, 1\}$ .*

*Proof of Theorem 6-1.*

**Lemma 6-2.** *There exist positive numbers  $C_9$  and  $C_{10}$  such that the following holds. For each  $N$  satisfying the conditions of Theorem I, there exist open subsets  $D_1, \dots, D_L$  of  $N$  such that following holds:*

- (1)  $L \leq C_9 V d^{-n}$ .
- (2) There exists  $L' \leq L$  such that

$$(i) \quad N^{(b)} \supset \bigcup_{i=1}^{L'} D_i \supset N^{(c)},$$

$$(ii) \quad \bigcup_{i=1}^L D_i \supset N^{(a)}$$

(3) For each  $\{i_1, \dots, i_k\} \subset \{1, 2, \dots, L\}$ , the set  $\cap_{j=1}^k D_{i_j}$  is contractible if it is nonempty.

(4) For each  $i \leq L$ , we have

$$\# \{j \leq L \mid D_i \cap D_j \neq \emptyset\} \leq C_{10}.$$

*Proof.* Let  $Z = \{p_1, p_2, \dots\}$  be a maximal subset of  $N^{(a/4)}$  such that  $d(p, q) \geq d/2$  holds for the two elements  $p$  and  $q$  of  $Z$ . A method similar to the proof of (1-1) in §1 shows that  $D_i = D_{p_i}(d)$  has the desired property.

Now we construct a simplicial complex  $\mathcal{S}_M$  as follows:

- (a) The vertex set of  $\mathcal{S}_M$  is  $\{1, 2, \dots, L\}$ .
- (b) For  $0 < i_1 < i_2 < \dots < i_k$ , the set  $\{i_1, \dots, i_k\}$  is the vertex set of some simplex of  $\mathcal{S}_M$  if and only if

$$D_{i_1} \cap D_{i_2} \cap \dots \cap D_{i_k} \neq \emptyset.$$

Denote by  $\mathcal{S}'_M$  the full subcomplex of  $\mathcal{S}_M$  whose vertex set is  $\{1, \dots, L'\}$ . The suffix  $M$  is omitted when no confusion arises. Let  $|\mathcal{S}|$  and  $|\mathcal{S}'|$  be geometric

realizations of  $\mathcal{S}$  and  $\mathcal{S}'$  respectively. It is well known that (3) in Lemma 6-2 implies that there exist homotopy equivalences

$$\Psi: |\mathcal{S}| \rightarrow \bigcup_{i=1}^L D_i \quad \text{and} \quad \Psi': |\mathcal{S}'| \rightarrow \bigcup_{i=1}^{L'} D_i$$

which commute with natural inclusions  $|\mathcal{S}'| \rightarrow |\mathcal{S}|$  and  $\bigcup_{i=1}^{L'} D_i \rightarrow \bigcup_{i=1}^L D_i$ .

Now, by (1) and (4) in Lemma 6-2, we see easily that  $\mathcal{S}$  satisfies the following conditions:

(i) The number of vertices is smaller than  $C_9 V d^{-n}$ .

(ii) There exist at most  $C_{10}$  vertices which can be joined with a given vertex of  $\mathcal{S}$  by some 1-simplex of  $\mathcal{S}$ .

We put  $C =$  the maximum integer smaller than  $C_9 V d^{-n}$ . Let  $Y$  be the set of all simplicial complexes satisfying conditions (i) and (ii) above.

**Lemma 6-3.** *There exists a constant  $C_{11}$  such that the number of isomorphism classes containing elements of  $Y$  is smaller than  $(C_{11} V d^{-n})^{C_{11} V d^{-n}}$ .*

*Proof.* Let  $\Delta^C$  denote the simplex whose vertex set is  $\{1, 2, \dots, C\}$ . It is easy to see that every element of  $Y$  can be represented as a subcomplex of  $\Delta^C$ .

Let  $Z$  be the set consisting of all subcomplexes of  $\Delta^C$  which have less than  $C_{10}$  vertices.

Define a map  $\Omega = (\omega_1, \omega_2, \dots, \omega_C): Y \rightarrow Z^C$ , as follows.

$\omega_i(\mathcal{S}) =$  the full subcomplex of  $\mathcal{S}$  whose vertex set is

$$\{j | j \text{ can be joined with } i \text{ by some 1-simplex of } \mathcal{S}.\}$$

Condition (ii) implies that  $\omega_i(\mathcal{S}) \in Z$ . Hence the map  $\Omega$  is well defined. (Here we identified an element of  $Y$  to a subcomplex of  $\Delta^C$  which is isomorphic to it.)

Clearly  $\Omega$  is injective. On the other hand, it can be proved by an easy combinatorial argument that there exists a positive number  $C_{12}$  depending only on  $n$ , the dimension, and satisfying

$$\#Z \leq D_{12} \cdot C^{C_{10}} \leq C_{12} \cdot (C_9 V d^{-n})^{C_{10}}.$$

Lemma 6-3 follows immediately from these two facts.

Now we return to the proof of Theorem 6-1. Since  $\mathcal{S}'$  is a full subcomplex of  $\mathcal{S}$  and since the order of the vertex set of  $\mathcal{S}$  is smaller than  $C$ , it follows that there exist at most  $2^C \cdot (C_{11} V d^{-n})^{C_{11} V d^{-n}}$  isomorphism classes containing pairs  $(\mathcal{S}_N, \mathcal{S}'_N)$  for some  $N$  satisfying the conditions of Theorem 1-1.

Therefore Theorem 1-1 follows immediately from the lemma below.

**Lemma 6-4.** *If  $(\mathcal{S}_N, \mathcal{S}'_N)$  is isomorphic to  $(\mathcal{S}_{N'}, \mathcal{S}'_{N'})$ , then  $N$  is homotopy equivalent to  $N'$ .*

*Proof.* Let  $G$  be an Abelian group and let  $\varphi: \pi_1(N) \rightarrow G$  be a homomorphism.  $G$  and  $\varphi$  induce local coefficient system on  $|\mathcal{S}|$  and  $|\mathcal{S}'|$ . Let  $G^\varphi$  denote this local coefficient system.

**Assertion 6-5.**  $\pi_1(N)$  is isomorphic to the image of the homomorphism:  $\pi_1(|\mathcal{S}'|) \rightarrow \pi_1(|\mathcal{S}|)$  induced by the inclusion map. Also, for each  $G$  and  $\varphi$ , the group  $H_*(N; G^\varphi)$  is isomorphic to the image of the homomorphism:  $H_*(|\mathcal{S}'|; G^\varphi) \rightarrow H_*(|\mathcal{S}|; G^\varphi)$  induced by the inclusion map.

*Proof.* Let  $\Phi$  be the map given in condition (3) in Theorem 1-1. Set  $N'' = \Phi^{-1}(\partial N' \times [0, 1/2]) \cup N'$ . Then the homomorphism:  $\pi_1(N'') \rightarrow \pi_1(N)$  is an isomorphism. Hence considering the diagram

$$\pi_1\left(\bigcup_{i=1}^{L'} D_i\right) \rightarrow \pi_1(N'') \rightarrow \pi_1\left(\bigcup_{i=1}^L D_i\right) \rightarrow \pi_1(N)$$

we can easily show that the homomorphism:  $\pi_1(\bigcup_{i=1}^L D_i) \rightarrow \pi_1(N)$  is injective on the image of the homomorphism:  $\pi_1(\bigcup_{i=1}^{L'} D_i) \rightarrow \pi_1(\bigcup_{i=1}^L D_i)$ .

On the other hand, since the homomorphism:  $\pi_1(N') \rightarrow \pi_1(N)$  is an isomorphism, considering the diagram

$$\pi_1(N') \rightarrow \pi_1\left(\bigcup_{i=1}^{L'} D_i\right) \rightarrow \pi_1(N),$$

we can easily show that the homomorphism:  $\pi_1(\bigcup_{i=1}^{L'} D_i) \rightarrow \pi_1(N)$  is surjective.

The above two facts immediately imply the statement on the fundamental groups. The proof of the statement on the homology groups is similar. The proof of Assertion 6-5 is completed.

Now we return to the proof of Lemma 6-4. Let  $\psi$  be the isomorphism between  $(\mathcal{S}_{N_1}, \mathcal{S}'_{N_1})$  and  $(\mathcal{S}_{N_2}, \mathcal{S}'_{N_2})$ . Let  $\varphi: N'_1 \rightarrow N_2$  be the composition of the following five maps: the injection:  $N'_1 \rightarrow \bigcup_{i=1}^{L'} D_i$ , the homotopy equivalence:  $\bigcup_{i=1}^{L'} D_i \rightarrow |\mathcal{S}_{N_1}|$ , the map  $|\psi|: |\mathcal{S}_{N_1}| \rightarrow |\mathcal{S}_{N_2}|$ , the homotopy equivalence:  $|\mathcal{S}_{N_2}| \rightarrow \bigcup_{i=1}^L D_i$  and the injection  $\bigcup_{i=1}^L D_i \rightarrow N_2$ . Then, using Assertion 6-5, it is easy to see that  $\varphi$  induces isomorphisms both on fundamental groups and on homology groups of any local coefficient system. Therefore  $\varphi$  is a homotopy equivalence, as desired. Thus the proofs of Lemma 6-4 and that of Theorem 6-1 are completed.

Next, we give an estimate on the number of homotopy types containing manifolds satisfying the conditions of Theorem I.

First we need an estimate on the number  $\varepsilon_1(a, D)$  in Theorem A. Heintze & Karcher [10, Corollary 2.3.2] gives

$$\varepsilon_1(a, D) \geq (2\pi a/\text{Vol}(S^m)) \cdot \sinh(D)^{-m+1}.$$

Therefore there exists a constant  $C_{13}$  such that

$$\varepsilon_1(a, D) \geq a \cdot \exp(-C_{13}D).$$

On the other hand, the number  $\varepsilon_3(V)$  given in Theorem 4-2 is

$$\varepsilon_1((\varepsilon_2/2)^n \text{Vol}(D^n), C_5(\varepsilon_2/2) \cdot V^{p_n}).$$



Therefore there exists a constant  $C_{14}$  such that

$$\varepsilon_3(V) \geq \exp(-C_{14}V^{p_n}).$$

On the other hand, we have proved in §4 the following: if we put  $a = \varepsilon_3(V)/64$ ,  $b = \varepsilon_3(V)/16$  and  $c = \varepsilon_3(V)/8$ , and if we assume that  $M$  satisfies the conditions of Theorem I, then  $M$  satisfies the conditions of Theorem 1-1.

Using these facts and Theorem 6-1, we obtain the following result.

**Theorem 6-6.** *For each positive integer  $n$  greater than 3, there exists a positive number  $C_{15}$  such that the following holds. For each positive number  $V$ , there exists at most  $\exp(\exp(C_{15}V^{p_n}))$  homotopy types containing manifolds satisfying the conditions of Theorem I.*

The number of homotopy types containing manifolds  $M$  satisfying  $c^+(M) \leq -\delta$  in addition, can be estimated by  $\exp(\exp(C_{16}(\delta)V))$ . (This fact can be proved by using Theorem 4-1' instead of Theorem 4-1 in the proof of Theorem 6-6.) Using this fact and Mostow's rigidity theorem ([14], [16]), we can easily prove Theorem II.

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